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THE ASYMPTOTIC DISTRIBUTION OF A SERIAL CORRELATION COEFFICIENT OF A LINEAR MARKOV PROCESS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled THE ASYMPTOTIC DISTRIBUTION OF A SERIAL CORRELATION COEFFICIENT OF A LINEAR MARKOV PROCESS submitted by JONATHAN BERKOWITZ in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The purpose of this thesis is to derive the maximum likelihood estimator of a parameter and to study the asymptotic distribution of the estimator based on the assumption that the sample observations are distributed according to a non-stationary normal Markov process when, in fact, the true multivariate model governing the distribution of the sample observations is a stationary normal Markov process. Daniels's saddlepoint approximation method using the method of steepest descent is reviewed and then used to obtain the asymptotic distribution of the estimator.

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CHAPTER I

INTRODUCTION

The maximum likelihood method of estimating a parameter is based on prior knowledge of the underlying distribution. The estimator is the function of the observed sample values that maximizes the joint density function of the sample (or the joint probability function for a discrete random variable). In theory, one can also obtain the distribution of the estimator from the assumed joint distribution of the observations. If the assumption of the underlying distribution is invalid, then inferences based on the derived distribution of the estimator will also be invalid. Such considerations are important, for example, in studying the power of hypothesis tests about the parameter.

The purpose of this thesis is to derive the maximum likelihood estimator $\hat{\beta}$ of a parameter β based on the assumption that the sample observations are distributed according to a non-stationary normal Markov process (Model I) and then to study the asymptotic distribution of $\hat{\beta}$ when, in fact, the true multivariate model governing the distribution of the sample observations is a stationary normal Markov process (Model II).

Model I

Suppose we have a sequence of random variables $\{X_o, X_1, \ldots, X_n\}$ such that X_o is normally distributed with mean a and variance σ_o^2, and $X_k | X_{k-1}$ is normally distributed with mean $\alpha + \beta X_{k-1}$ and variance σ^2 for $k = 1,2,\ldots,n$.

Assume the normal 'scores' X_k are Markovian, that is,

$$f(x_k|x_{k-1},x_{k-2},\ldots,x_o) = f(x_k|x_{k-1}) \quad .$$

Then

$$\begin{aligned} f(x_o,\ldots,x_n) &= f(x_o)\cdot f(x_1,\ldots,x_n|x_o) \\ &= f(x_o)\cdot f(x_1|x_o)\cdot f(x_2,\ldots,x_n|x_1,x_o) \\ &= f(x_o)\cdot f(x_1|x_o)\cdot f(x_2|x_1,x_o)\cdot f(x_3,\ldots,x_n|x_o,x_1,x_2) \\ &= f(x_o)\cdot f(x_1|x_o)\cdot f(x_2|x_1)\cdot f(x_3,\ldots,x_n|x_2) \\ &\ \ \vdots \\ &= f(x_o)\cdot f(x_1|x_o)\cdot f(x_2|x_1)\cdots f(x_n|x_{n-1}) \end{aligned}$$

where $f(x_o,x_1,\ldots,x_n)$ is the joint probability density function.

We have

$$f(x_o) = \frac{1}{\sigma_o\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_o - a}{\sigma_o}\right)^2\right]$$

and

$$f(x_k|x_{k-1}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_k - \alpha - \beta x_{k-1}}{\sigma}\right)^2\right] \quad .$$

therefore,

$$(1.1) \qquad f(x_o,\ldots,x_n) = \frac{1}{\sigma_o\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_o - a}{\sigma_o}\right)^2\right]\cdot \frac{1}{\sigma^n(2\pi)^{n/2}}$$

$$\times \exp\left[-\frac{1}{2}\sum_{j=1}^{n}\left(\frac{x_j - \alpha - \beta x_{j-1}}{\sigma}\right)^2\right] \quad .$$

Next, under Model I, we obtain the maximum likelihood estimator $\hat{\beta}$ for the parameter β. The likelihood function is defined to be the joint density function evaluated at $x_o, x_1, \ldots, x_n$, that is,

$$L(\alpha,\beta) = f(x_o,\ldots,x_n) \quad ,$$

where $f(x_o,\ldots,x_n)$ is given by equation (1.1).

So

$$\ln L(\alpha,\beta) = K - \frac{1}{2}\left(\frac{x_o - a}{\sigma_o}\right)^2 - \frac{1}{2}\sum_{j=1}^{n}\left(\frac{x_j - \alpha - \beta x_{j-1}}{\sigma}\right)^2 \quad .$$

Then the partial derivatives with respect to α and β are:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{1}{\sigma^2}\sum_{j=1}^{n}(x_j - \alpha - \beta x_{j-1}) \quad ,$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{1}{\sigma^2}\sum_{j=1}^{n}(x_j - \alpha - \beta x_{j-1})(x_{j-1}) \quad .$$

Setting the partial derivatives equal to zero and denoting the maximum likelihood estimates of α and β by $\hat{\alpha}$ and $\hat{\beta}$ respectively, gives:

$$\sum_{j=1}^{n} x_j = n\hat{\alpha} + \hat{\beta}\sum_{j=1}^{n} x_{j-1} \quad ,$$

$$\sum_{j=1}^{n} x_j x_{j-1} = \hat{\alpha}\sum_{j=1}^{n} x_{j-1} + \hat{\beta}\sum_{j=1}^{n} x_{j-1}^2 \quad .$$

To solve for $\hat{\alpha}$ and $\hat{\beta}$, multiply the above equations by $\sum x_{j-1}$ and n respectively. Subtracting the resulting first equation from the second gives

$$n\hat{\beta}\sum_{j=1}^{n} x_{j-1}^2 - \hat{\beta}\left(\sum_{j=1}^{n} x_{j-1}\right)^2 = n\sum_{j=1}^{n} x_j x_{j-1} - \left(\sum_{j=1}^{n} x_j\right)\left(\sum_{j=1}^{n} x_{j-1}\right) \quad .$$

Thus

$$(1.2)\qquad \hat{\beta} = \frac{n\sum_{j=1}^{n} x_j x_{j-1} - (\sum_{j=1}^{n} x_j)(\sum_{j=1}^{n} x_{j-1})}{n\sum_{j=1}^{n} x_{j-1}^2 - (\sum_{j=1}^{n} x_{j-1})^2}$$

$$= \frac{\sum_{j=1}^{n} (x_j - \bar{x}_u)(x_{j-1} - \bar{x}_l)}{\sum_{j=1}^{n} (x_{j-1} - \bar{x}_l)^2}$$

where

$$\bar{x}_l = \frac{x_o + x_1 + \ldots + x_{n-1}}{n}$$

and

$$\bar{x}_u = \frac{x_1 + x_2 + \ldots + x_n}{n} \quad .$$

Solving for $\hat{\alpha}$ gives

$$\hat{\alpha} = \bar{x}_u - \hat{\beta}\, \bar{x}_l \quad .$$

Model II

Now suppose that, in fact, $\{X_o, X_1, \ldots, X_n\}$ obey a stationary linear normal Markov process

$$X_i = \beta X_{i-1} + e_i \quad , \quad i = 0,1,\ldots,n \quad , \quad |\beta| < 1,$$

where $\{e_o, e_1, \ldots, e_n\}$ are independent and identically distributed standard normal random variables. (See Daniels (5), p. 176.) Note that there is no loss in generality in assuming the common variance σ^2 of the residuals to be unity since $\hat{\beta}$ is unchanged if each x_i is replaced with x_i/σ. The parameter β is called the serial correlation coefficient.

The condition $|\beta| < 1$ is necessary for the stationarity of the process, a result that will be shown later. Then the joint probability distribution of $\{X_o,\ldots,X_n\}$ is given by

$$(1.3) \qquad dF(x_o,\ldots,x_n) = \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}} \frac{1}{|\underline{C}|^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}\,\underline{x}'\underline{C}^{-1}\underline{x}\}\, dx_o \ldots dx_n$$

where $\underline{C}$ is the variance-covariance matrix and $\underline{x}' = (x_o,\ldots,x_n)$.

Early work on the problem of serial correlation was done by Yule (17) in 1921. It was shown by both Yule and Bartlett (4) that the ordinary tests of significance become invalid if successive observations are not independent of one another. Wold's (16) pioneer work in time series analysis also created interest in serial correlation.

However, not until a 'circular' definition of the serial correlation coefficient was suggested by Hotelling, was much progress made. R. L. Anderson (3) obtained the exact distribution of r, where

$$r = \frac{x_1x_2 + \ldots + x_nx_1 - (\sum x_i)^2/n}{\sum x_i^2 - (\sum x_i)^2/n}$$

is an estimate of the serial correlation coefficient. Since the exact distributions of such statistics are complicated, simple approximations were sought. Using various smoothing techniques, Koopmans (9) and Dixon (6) found such approximate distributions for the circular coefficient with known mean, the equivalence of which was shown by Rubin (15).

Koopmans (9) considered the stochastic process

$$X_i = \beta X_{i-1} + e_i \quad, \; i = 1,2,\ldots,n \quad, \; |\beta| < 1,$$

where β is the serial correlation coefficient and the $\{z_t\}$ are independent $N(0,\sigma^2)$ random variables. For testing the hypothesis that $\beta = 0$, Koopmans showed that it is sufficient to know the distribution of

$$\frac{\sum_{i=1}^{n-1} (x_i - \bar{x})(x_{i+1} - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} .$$

Dixon's (6) work in the area was concerned with the moments of the ratio

$$\frac{\delta_n^2}{V_n} = \frac{\sum_{i=1}^{n} (x_{i+1} - x_i)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

and the approximate distribution and moments of

$$\left(2 - \frac{\delta_n^2}{V_n}\right)^2 .$$

Anderson's results were extended by Madow (11) to the circular case where $\beta \neq 0$ and an exact distribution was found.

Madow's method was employed by Leipnik(10) to deal with the case of the circular Markov process. An approximate distribution was found by smoothing the joint characteristic function of the numerator and denominator of the sample correlation coefficient; that is, by summing over a discrete set of roots using an approximating integral.

Until Daniels's (5) work, the non-circular cases had not been dealt with to any great extent. For both circular and non-circular Markov processes with known and unknown means, Daniels used the method of steepest descents to obtain the approximate distribution of the sample

serial correlation coefficient r. (The method is discussed by Jeffreys and Jeffreys (7)). His estimate of β in the non-circular case with known mean is

$$r = \frac{c}{c_o} = \frac{x_1x_2 + \ldots + x_{n-1}x_n}{\frac{1}{2}x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + \frac{1}{2}x_n^2} .$$

If the mean is unknown, his estimate is

$$r = \frac{c - (n-1)\bar{x}^2}{c_o - (n-1)\bar{x}^2} = \frac{(x_1-\bar{x})(x_2-\bar{x}) + \ldots + (x_{n-1}-\bar{x})(x_n-\bar{x})}{\frac{1}{2}(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \ldots + (x_{n-1}-\bar{x})^2 + \frac{1}{2}(x_n-\bar{x})^2} .$$

Patton (14) also considered a non-circular Markov process, both with known and unknown means. With known mean, the estimate of the serial correlation coefficient was

$$r = \frac{c}{c_o} = \frac{\frac{1}{2}x_1^2 + x_1x_2 + \ldots + x_{n-1}x_n + \frac{1}{2}x_n^2}{x_1^2 + \ldots + x_n^2} .$$

If the mean is unknown,

$$r = \frac{c - n\bar{x}^2}{c_o - n\bar{x}^2} = \frac{\frac{1}{2}(x_1-\bar{x})^2+(x_1-\bar{x})(x_2-\bar{x})+\ldots+(x_{n-1}-\bar{x})(x_n-\bar{x})+\frac{1}{2}(x_n-\bar{x})^2}{(x_1-\bar{x})^2+\ldots+(x_n-\bar{x})^2} .$$

Daniels obtained the following results in the non-circular case. With known mean the approximate distribution of the estimate r is

$$h(r) = \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}n-1)}{\Gamma(\frac{1}{2}n-\frac{1}{2})} \frac{(1-\beta^2)^{\frac{1}{2}}(1-r^2)^{\frac{1}{2}n-1}}{(1-\beta r)(1-2\beta r+\beta^2)^{\frac{1}{2}n-1}} (1 + O(n^{-1})) .$$

In Leipnik's form, where the approximation has remainder that is relatively $O(n^{-3/2})$, the distribution may be written as

$$h(r) = \frac{\Gamma(\frac{1}{2}N+1)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}N+\frac{1}{2})} \frac{(1-r^2)^{\frac{1}{2}(N-1)}}{(1-2\beta r+\beta^2)^{\frac{1}{2}N}} (1 + O(n^{-3/2}))$$

with

$$N = n - 1 + \frac{\beta}{1-\beta^2} \quad .$$

Patton's results are summarized below. With known mean,

$$h(r) = \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}n-1)}{\Gamma(\frac{1}{2}n-\frac{1}{2})} \frac{(1-\beta^2)^{\frac{1}{2}}(1-r^2)^{\frac{1}{2}n}}{(1-2\beta r+\beta^2)^{\frac{1}{2}n-1}(1-\beta r)(1-r)(1+\beta)} (1 + O(n^{-1})) \;,$$

or, in Leipnik's form,

$$h(r) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}N+2)}{\Gamma(\frac{N+1}{2})\left(N(1+\beta)+2\right)} \frac{(1-r^2)^{\frac{1}{2}(N+1)}}{(1-r)(1-2\beta r+\beta^2)^{\frac{1}{2}N}} (1 + O(n^{-3/2}))$$

with

$$N = n - 1 + \frac{\beta^2}{1-\beta^2} \quad .$$

If the mean is unknown,

$$h(r) = \frac{n}{2\sqrt{\pi}} \frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{1}{2}n-1)} \frac{(1-r^2)^{\frac{1}{2}(n-1)}}{(1-\beta^2)^{\frac{1}{2}}(1-\beta r)(1-2\beta r+\beta^2)^{\frac{1}{2}(n-3)}} (1 + O(n^{-1})) \;,$$

or, in Leipnik's form,

$$h(r) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{N+5}{2})}{\Gamma(\frac{1}{2}N+1)\left((1-\beta^2)(N-1)+4\right)} \frac{(1-r^2)^{\frac{1}{2}N}}{(1-2\beta r+\beta^2)^{\frac{1}{2}(N-1)}} (1 + O(n^{-3/2}))$$

with

$$N = n - 1 + \frac{\beta^2}{1-\beta^2} \quad .$$

In Chapter II, we derive the asymptotic distribution of $\hat{\beta}$ following the methods of Daniels, and assuming that the observations are distributed according to Model II.

CHAPTER II

THE ASYMPTOTIC DISTRIBUTION OF A SERIAL CORRELATION COEFFICIENT OF A LINEAR MARKOV PROCESS

2.1 Daniels's Saddlepoint Approximation Method

The following results are discussed in Daniels's (5) paper of 1956. We wish to explore the distribution of statistics of the form

$$r = \frac{c}{c_o} \quad ,$$

where c_o is almost surely positive. Let c_o, c have the joint probability density $f(c_o,c)$. To find the distribution of r we use the transformation $c = r\,c_o$, $c_o = c_o$. The Jacobian of such a transformation is

$$\left|\frac{\partial(c_o,c)}{\partial(c_o,r)}\right| = c_o \quad .$$

So the joint probability density of c_o and r is

$$c_o\ f(c_o,\ r\ c_o) \quad .$$

Then the probability density of r can be expressed as a marginal density by integrating out c_o, giving

$$\text{(2.1.1)} \qquad h(r) = \int_0^{\infty} c_o\ f(c_o,\ r\ c_o)\ dc_o \quad .$$

Let $M(t_o,t) = E(\exp\,(t_o c_o + tc))$

$$= \int_{-\infty}^{\infty}\int_0^{\infty} \exp(t_o c_o + tc)\ f(c_o,c)\ dc_o dc$$

be the joint moment-generating function of c_o and c. Then by the

Fourier inversion formula, we have

$$f(c_o,c) = \frac{1}{(2\pi i)^2} \int\int M(t_o,t) \exp\left(-(t_o c_o + tc)\right) dt_o dt \quad .$$

The integration is along the imaginary axes of t_o and t (that is, in the t_o and t planes from $+i\infty$ to $-i\infty$), or along any allowable deformation of these paths. In other words, the paths of integration are along any paths from $\xi-i\infty$ to $\xi+i\infty$ such that no singularities of $M(t_o,t)$ lie on the new paths of integration or between them and the imaginary axes. Since $c = rc_o$,

$$f(c_o,rc_o) = \frac{1}{(2\pi i)^2} \int\int M(t_o,t) \exp\left(-c_o(t_o + rt)\right) dt_o dt \ . \tag{2.1.2}$$

Using the linear transformation $u = t_o + r\,t$, $t = t$,with the Jacobian

$$\left| \frac{\partial(t_o,t)}{\partial(u,t)} \right| = 1 \quad ,$$

equation (2.1.2) becomes

$$f(c_o,rc_o) = \frac{1}{(2\pi i)^2} \int\int M(u - rt,t) \exp(-c_o u)\, dudt \quad .$$

The integration of u is taken over a path in the u-plane corresponding to that of t_o in the t_o-plane.

Then

$$\int_0^\infty f(c_o,rc_o)\, e^{c_o u}\, dc_o \tag{2.1.3}$$

$$= \int_0^\infty \left[\frac{1}{(2\pi i)^2} \int\int M(u - rt,t)\, e^{-c_o u}\, du\, dt \right] e^{c_o u}\, dc_o$$

$$= \frac{1}{2\pi i} \int \left[\int_0^\infty \left[\frac{1}{2\pi i} \int M(u - rt,t)\ e^{-c_o u}\ du \right] e^{c_o u}\ dc_o \right] dt\ .$$

The usual Fourier inversion formula states

$$A(v) = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty A(w)\ \exp\left(i(w - v)c_o \right) dwdc_o$$

or equivalently,

$$M(u - rt,t) = \int_0^\infty \left(\frac{1}{2\pi i} \int M(u - rt,t)\ e^{-c_o u} du \right) e^{c_o u}\ dc_o\ .$$

Thus from equation (2.1.3)

$$\int_0^\infty f(c_o, rc_o)\ e^{c_o u}\ dc_o = \frac{1}{2\pi i} \int M(u - rt,t)\ dt\ .$$

When differentiation under the integral sign with respect to u is permissible, we obtain

$$\text{(2.1.4)} \qquad \int_0^\infty f(c_o, rc_o) c_o e^{c_o u}\ dc_o = \frac{1}{2\pi i} \int \frac{\partial}{\partial u} \left(M(u - rt,t) \right) dt\ .$$

Letting $u = 0$, the left-hand side of equation (2.1.4) becomes $h(r)$, so we have

$$h(r) = \frac{1}{2\pi i} \int \frac{\partial}{\partial u} \left(M(u - rt,t) \right) \Big|_{u=0}\ dt\ .$$

It is often desirable to transform t to another variable z by $t = t(z,u)$ where $t(z,0)$ maps the t-plane onto some region of the z-plane. Then

$$M(u - rt,t)dt = M\left(u - rt(z,u), t(z,u)\right) \frac{\partial t(z,u)}{\partial z}\ dz\ .$$

For notational ease, we write this as

$$M(u - rt,t)\,\frac{\partial t}{\partial z}\,dz \quad .$$

Thus

$$\int_0^\infty f(c_o,rc_o)\,e^{c_o u}\,dc_o = \frac{1}{2\pi i}\int M(u - rt,t)\,\frac{\partial t}{\partial z}\,dz \quad .$$

The integration is now along the transformed contour in the z-plane. Differentiating once again with respect to u under the integral sign and letting $u = 0$ gives

$$\text{(2.1.5)} \qquad h(r) = \frac{1}{2\pi i}\int \frac{\partial}{\partial u}\left(M(u - rt,t)\,\frac{\partial t}{\partial z}\right)\Big|_{u=0}\,dz \quad .$$

This equation is known as the Cramer-Geary inversion formula and will be used in the work to follow.

It is desirable to give an approximate evaluation of integrals of the above type when moderately large samples are used to calculate the statistics c and c_o. In the cases considered it has been found that the integrands can be written in the form $\phi(z)\big(\psi(z)\big)^n$ where n is the sample size.

The method of steepest descents is used. (See Jeffreys and Jeffreys (7).) Choose a contour that passes through a saddlepoint $\hat{z}$ at which $\psi'(z) = 0$. As n increases, the major contribution to the integral comes from a small neighbourhood around $\hat{z}$. Now the integrand is expanded about $\hat{z}$ and term by term integration gives an asymptotic expansion in powers of (n^{-1}). The dominant term is called the saddlepoint approximation.

2.2 Linear Markov Processes

Consider a subset of time points $t_1 < t_2 < \ldots < t_n$, and the random variables $X(t_j)$ at time t_j. The joint distribution of $X(t_1), \ldots, X(t_n)$ is denoted by $p\{X(t_1), \ldots, X(t_n)\}$. The process is called completely stationary if

$$p\{X(t_1), \ldots, X(t_n)\} = p\{X(t_1 - \tau), \ldots, X(t_n - \tau)\}$$

for all subsets $t_1 < t_2 < \ldots < t_n$ and all τ. In other words, the joint distribution is invariant under a change of time origin. A process is stationary of order k if

$$E\{X^{\alpha_1}(t_1) \cdot \cdot \cdot X^{\alpha_n}(t_n)\} = E\{X^{\alpha_1}(t_1 - \tau) \cdot \cdot \cdot X^{\alpha_n}(t_n - \tau)\}$$

for all τ and all $\alpha_1 + \ldots + \alpha_n \leq k$.

If the process is stationary of order one,

$$E\{X(t)\} = E\{X(t - \tau)\} = \mu \quad ,$$

that is, the mean is independent of time. If the process is stationary of order two,

$$E\{X^2(t)\} = E\{X^2(t - \tau)\} \quad \text{for all } \tau, \text{ and}$$

$$E\{X(t_1)X(t_2)\} = E\{X(t_1 - \tau)X(t_2 - \tau)\} \text{ for all } \tau.$$

Then

$$\begin{aligned} \mathrm{Var}(X(t)) &= E\{X^2(t)\} - \{E(X(t))\}^2 \\ &= \mathrm{Var}\{X(t - \tau)\} \\ &= \sigma^2 \text{ for all } \tau, \end{aligned}$$

that is, variance is independent of time.

Also,

$$\mathrm{Cov}\big(X(t_1),X(t_2)\big) = E\big(X(t_1)X(t_2)\big) - E\big(X(t_1)\big)E\big(X(t_2)\big)$$

$$= \mathrm{Cov}\big(X(t_1 - \tau),X(t_2 - \tau)\big)$$

$$= C(t_2 - t_1) \quad \text{for all} \quad \tau .$$

$C(t_2 - t_1)$ is called the autocovariance and is independent of the time origin. Define the autocorrelation function by

$$\rho(\tau) = \frac{C(t_2 - t_1)}{\sigma^2}$$

where $\tau = t_2 - t_1$ is the lag.

Consider a linear Markov process defined by

$$X_j = \beta X_{j-1} + e_j \quad .$$

If we assume that the initial state is X_o, then

$$X_j = \beta^j X_o + \beta^{j-1} e_1 + \beta^{j-2} e_2 + \ldots + \beta^2 e_{j-2} + \beta e_{j-1} + e_j \quad .$$

Since $E(e_j) = 0$,

$$E(X_j) = \beta^j X_o \to 0 \quad \text{as} \quad j \to \infty \quad \text{if} \quad |\beta| < 1 \quad ,$$

that is, for the process to become stationary to the first order, it is necessary that $|\beta| < 1$.

Since $E(e_j^2) = \mathrm{Var}(e_j) = 1$ and $E(e_i e_j) = 0 \quad (i \neq j)$, we have

$$\mathrm{Var}(X_j) = E(X_j^2) - \big(E(X_j)\big)^2$$

$$= \left(\frac{1 - \beta^{2j}}{1 - \beta^2} + \beta^{2j} X_o^2\right) - (\beta^j X_o)^2$$

$$= \frac{1 - \beta^{2j}}{1 - \beta^2} \quad .$$

As $j \to \infty$, $\mathrm{Var}(X_j) \to \dfrac{1}{1 - \beta^2}$ if $|\beta| < 1$.

For any fixed $\tau > 0$, and $j > \tau$,

$$\mathrm{Cov}(X_j, X_{j-\tau}) = E(X_j X_{j-\tau}) - E(X_j)E(X_{j-\tau})$$

$$= \left[\beta^{\tau} \cdot \left(\frac{1 - \beta^{2(j-\tau)}}{1 - \beta^2}\right) + \beta^{2j-\tau} X_o^2\right] - (\beta^j X_o)(\beta^{j-\tau} X_o)$$

$$= \beta^{\tau} \cdot \left(\frac{1 - \beta^{2(j-\tau)}}{1 - \beta^2}\right) .$$

As $j \to \infty$,

(2.2.1) $$\mathrm{Cov}(X_j, X_{j-\tau}) \to \frac{\beta^{\tau}}{1 - \beta^2} \quad \text{if} \quad |\beta| < 1.$$

Also, for any fixed $\tau < 0$, $\tau = -|\tau|$ and

$$\mathrm{Cov}(X_j, X_{j-\tau}) = \mathrm{Cov}(X_j, X_{j+|\tau|})$$

$$= \beta^{|\tau|} \cdot \left(\frac{1 - \beta^{2j}}{1 - \beta^2}\right) .$$

As $j \to \infty$,

(2.2.2) $$\mathrm{Cov}(X_j, X_{j-\tau}) \to \frac{\beta^{|\tau|}}{1 - \beta^2} \quad \text{if} \quad |\beta| < 1.$$

Therefore, if $|\beta| < 1$, the process approaches stationarity of the second order as time advances indefinitely. Since a multivariate normal distribution is determined by its first two moments, a normal process which is stationary to the second order is completely stationary. In subsequent sections we assume that the process has reached a completely stationary state.

2.3 The Joint Distribution of $X_o, \ldots, X_n$

Recall Model II: If $X_o, \ldots, X_n$ form the observed sample from the stationary linear Markov process then they have a joint multivariate normal distribution since $e_1, \ldots, e_n$ are independent standard normal random variables.

Using the result of equations (2.2.1) and (2.2.2)

$$\mathrm{Cov}(X_j, X_{j-\tau}) = \frac{\beta^{|\tau|}}{1-\beta^2} \quad \text{if} \quad |\beta| < 1 ,$$

the variance-covariance matrix of $\underline{x}' = (x_o, x_1, \ldots, x_n)$ is

$$\underline{C} = \begin{bmatrix} \frac{1}{1-\beta^2} & \frac{\beta}{1-\beta^2} & \cdots & \frac{\beta^n}{1-\beta^2} \\ \frac{\beta}{1-\beta^2} & \frac{1}{1-\beta^2} & \cdots & \frac{\beta^{n-1}}{1-\beta^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta^n}{1-\beta^2} & \frac{\beta^{n-1}}{1-\beta^2} & \cdots & \frac{1}{1-\beta^2} \end{bmatrix} .$$

Next, evaluate $|\underline{C}|$ and C^{-1}.

$$|\underline{C}| = \begin{vmatrix} \frac{1}{1-\beta^2} & \frac{\beta}{1-\beta^2} & \cdots & \frac{\beta^n}{1-\beta^2} \\ \frac{\beta}{1-\beta^2} & \frac{1}{1-\beta^2} & \cdots & \frac{\beta^{n-1}}{1-\beta^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta^n}{1-\beta^2} & \frac{\beta^{n-1}}{1-\beta^2} & \cdots & \frac{1}{1-\beta^2} \end{vmatrix}_{n+1} .$$

Multiply the second column by β and subtract from the first column.

$$
|\underline{C}| = \begin{vmatrix} 1 & \frac{\beta}{1-\beta^2} & \cdots & \frac{\beta^n}{1-\beta^2} \\ 0 & \frac{1}{1-\beta^2} & \cdots & \frac{\beta^{n-1}}{1-\beta^2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{\beta^{n-1}}{1-\beta^2} & \cdots & \frac{1}{1-\beta^2} \end{vmatrix}_{n+1}
$$

$$
= \begin{vmatrix} \frac{1}{1-\beta^2} & \frac{\beta}{1-\beta^2} & \cdots & \frac{\beta^{n-1}}{1-\beta^2} \\ \frac{\beta}{1-\beta^2} & \frac{1}{1-\beta^2} & \cdots & \frac{\beta^{n-2}}{1-\beta^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta^{n-1}}{1-\beta^2} & \frac{\beta^{n-2}}{1-\beta^2} & \cdots & \frac{1}{1-\beta^2} \end{vmatrix}_{n} .
$$

Continuing in the same manner gives

$$
(2.3.1) \qquad |\underline{C}| = \begin{vmatrix} \frac{1}{1-\beta^2} & \frac{\beta}{1-\beta^2} \\ \frac{\beta}{1-\beta^2} & \frac{1}{1-\beta^2} \end{vmatrix} = \frac{1}{1-\beta^2} .
$$

To find $\underline{C}^{-1}$, recall that $\underline{C}^{-1} = (c_{jk}) = \left(\frac{\text{adj } C_{kj}}{|\underline{C}|} \right)$.

Obtain adj C_{kj} in the same manner as $|\underline{C}|$. By the symmetry of $\underline{C}$, $c_{jk} = c_{kj}$ for $j \neq k$. Since $|\underline{C}| = \frac{1}{1-\beta^2}$, by equation (2.3.1), it follows that $\underline{C}^{-1}$ is the $(n\times n)$ matrix

$$\underline{C}^{-1} = \begin{bmatrix} 1 & -\beta & & & & & \\ -\beta & 1+\beta^2 & -\beta & & & (0) & \\ & -\beta & 1+\beta^2 & -\beta & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (0) & & & -\beta & 1+\beta^2 & -\beta \\ & & & & & -\beta & 1 \end{bmatrix}.$$

2.4 The Asymptotic Distribution of $\hat{\beta}$

For Model II the joint distribution of $x_o, x_1, \ldots, x_n$ is

$$dF(x_o, x_1, \ldots, x_n) = \left(\frac{1}{2\pi}\right)^{\frac{n+1}{2}} \frac{1}{|C|^{\frac{1}{2}}} \exp\left(\frac{-1}{2} \underline{x}'\underline{C}^{-1}\underline{x} \right) dx_o \ldots dx_n$$

where $\underline{C}$ is the given variance-covariance matrix and $\underline{x}' = (x_o, x_1, \ldots, x_n)$. Writing $\hat{\beta}$ as $\hat{\beta} = \frac{c}{c_o}$, where c is the numerator of (1.2) and c_o is the denominator of (1.2), the joint moment generating function of c and c_o is

$$M(t_o,t) = E\left(\exp(t_o c_o + tc)\right)$$

$$= \left(\frac{(1-\beta^2)}{(2\pi)^{n+1}}\right)^{1/2} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp(t_o c_o + tc - \frac{1}{2}\underline{x}'\underline{C}^{-1}\underline{x})\, dx_o \ldots dx_n \quad .$$

Writing

$$t_o c_o + tc - \frac{1}{2}\underline{x}'\underline{C}^{-1}\underline{x} = -\frac{1}{2}\underline{x}'\underline{B}\,\underline{x} \quad ,$$

where $\underline{B}$ is the appropriate $(n+1)\times(n+1)$ matrix, gives

$$M(t_o,t) = \left(\frac{(1-\beta^2)}{(2\pi)^{n+1}}\right)^{1/2} \int\cdots\int \exp(-\frac{1}{2}\underline{x}'\underline{B}\,\underline{x})\, dx_o \ldots dx_n \quad .$$

Diagonalizing $\underline{B}$ with the transformation having the Jacobian $|\underline{B}|^{-\frac{1}{2}}$ gives

$$\text{(2.4.1)} \qquad M(t_o,t) = \frac{(1-\beta^2)^{\frac{1}{2}}}{\left||\underline{B}|\right|^{\frac{1}{2}}} \quad .$$

Next, we need to define $\underline{B}$ and evaluate $|\underline{B}|$.

Recall, from (1.2),

$$\hat{\beta} = \frac{c}{c_o} = \frac{n\sum_{j=1}^{n} x_j x_{j-1} - (\sum_{j=1}^{n} x_j)(\sum_{j=1}^{n} x_{j-1})}{\sum_{j=1}^{n} x_{j-1}^2 - (\sum_{j=1}^{n} x_{j-1})^2} \quad .$$

Thus, $\quad t_o c_o + tc - \frac{1}{2}\,\underline{x}'\underline{C}^{-1}\underline{x}$

$$= t_o n \sum_{j=1}^{n} x_{j-1}^2 - t_o\left(\sum_{j=1}^{n} x_{j-1}\right)^2 + tn\sum_{j=1}^{n} x_j x_{j-1} - t\left(\sum_{j=1}^{n} x_j\right)\left(\sum_{j=1}^{n} x_{j-1}\right)$$

$$- \frac{1}{2}(x_o, x_1, \ldots, x_n)\begin{bmatrix} 1 & -\beta & & & & & \\ -\beta & 1+\beta^2 & -\beta & & & (0) & \\ & -\beta & 1+\beta^2 & -\beta & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & (0) & & & -\beta & 1+\beta^2 & -\beta \\ & & & & & -\beta & 1 \end{bmatrix}\begin{bmatrix} x_o \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

$$= -\frac{1}{2}\Big\{ -2tn(x_o x_1 + x_1 x_2 + \ldots + x_{n-1}x_n) - 2t_o n(x_o^2 + x_1^2 + \ldots + x_{n-1}^2)$$

$$+ 2t(x_1 + \ldots + x_n)(x_o + \ldots + x_{n-1}) + 2t_o(x_o + x_1 + \ldots + x_{n-1})^2$$

$$+ x_o^2 + x_n^2 + (1+\beta^2)(x_1^2 + x_2^2 + \ldots + x_{n-1}^2)$$

$$- 2\beta(x_o x_1 + x_1 x_2 + \ldots + x_{n-2}x_{n-1} + x_{n-1}x_n)\Big\}$$

$= -\frac{1}{2}\,\underline{x}'\underline{B}\,\underline{x}$, where $\underline{B}$ is given by

$$\underline{B} = \begin{bmatrix} a & c & f & f & f & \cdot & \cdot & \cdot & f & f & i \\ c & b & e & d & d & \cdot & \cdot & \cdot & d & d & i \\ f & e & b & e & d & \cdot & \cdot & \cdot & d & d & i \\ f & d & e & b & e & \cdot & \cdot & \cdot & d & d & i \\ f & d & d & e & b & \cdot & \cdot & \cdot & d & d & i \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f & d & d & d & d & \cdot & \cdot & \cdot & b & e & i \\ f & d & d & d & d & \cdot & \cdot & \cdot & e & b & g \\ i & i & i & i & i & \cdot & \cdot & \cdot & i & g & l \end{bmatrix}_{(n+1)\times(n+1)} \quad (2.4.2)$$

with:

$$a = 1 - 2t_o n + 2t_o$$

$$b = 1 + \beta^2 - 2t_o n + 2t_o + 2t$$

$$c = -\beta - tn + t + 2t_o$$

$$d = 2t_o + 2t$$

$$e = -\beta - tn + 2t + 2t_o$$

$$f = 2t_o + t$$

$$g = -\beta - tn + t$$

$$i = t \quad .$$

To evaluate the determinant $|\underline{B}|$, the bordering technique of Aitken (2) and Muir (12) will be used. The method consists of adding another row and column to the matrix without altering the value of the determinant. On the determinant

$$|\underline{B}| = \begin{vmatrix} 1 & \frac{f}{d} & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \frac{i}{d} \\ 0 & a & c & f & f & f & \cdot & \cdot & \cdot & f & f & i \\ 0 & c & b & e & d & d & \cdot & \cdot & \cdot & d & d & i \\ 0 & f & e & b & e & d & \cdot & \cdot & \cdot & d & d & i \\ 0 & f & d & e & b & e & \cdot & \cdot & \cdot & d & d & i \\ 0 & f & d & d & e & b & \cdot & \cdot & \cdot & d & d & i \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot \\ 0 & f & d & d & d & d & \cdot & \cdot & \cdot & b & e & i \\ 0 & f & d & d & d & d & \cdot & \cdot & \cdot & e & b & g \\ 0 & i & i & i & i & i & \cdot & \cdot & \cdot & i & g & 1 \end{vmatrix}$$

perform the following operations:

(1) Multiply Row 1 by (-d) and add to Row k, for k = 3,...,n+1;

(2) Multiply Row 1 by (-i) and add to Row n+2;

(3) Multiply Row 1 by (-f) and add to Row 2.

Thus we obtain

$$|\underline{B}| = \begin{vmatrix} 1 & \frac{f}{d} & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \frac{i}{d} \\ -f & a-\frac{f^2}{d} & c-f & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & i-\frac{if}{d} \\ -d & c-f & b-d & e-d & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ -d & 0 & e-d & b-d & e-d & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ -d & 0 & 0 & e-d & b-d & e-d & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ -d & 0 & 0 & 0 & e-d & b-d & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot \\ -d & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & b-d & e-d & 0 \\ -d & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & e-d & b-d & g-i \\ -i & i-\frac{if}{d} & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & g-i & 1-\frac{i^2}{d} \end{vmatrix} .$$

Now let

$$c - f = -(\beta + tn) = q \quad , \text{ and}$$

$$b - d = 1 + \beta^2 - 2t_o n = p \ ,$$

and then note that

$$e - d = -(\beta + tn) = q \quad ,$$

$$g - i = -(\beta + tn) = q \quad , \text{ and}$$

$$i - \frac{if}{d} = \frac{i^2}{d} \quad .$$

Then

$$|\underline{B}| = \begin{vmatrix}
1 & \frac{f}{d} & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \frac{i}{d} \\
-f & a-\frac{f^2}{d} & q & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & \frac{i^2}{d} \\
-d & q & p & q & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
-d & 0 & q & p & q & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
-d & 0 & 0 & q & p & q & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
-d & 0 & 0 & 0 & q & p & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot \\
-d & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & p & q & 0 \\
-d & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & q & p & q \\
-i & \frac{i^2}{d} & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & q & 1-\frac{i^2}{d}
\end{vmatrix}$$

The row totals from Column 2 to Column n+2 inclusive are found to be

$$n - 1 + \frac{f}{d} + \frac{i}{d} = n \quad ,$$

$$a - \frac{f^2}{d} + \frac{i^2}{d} + q = q + 1 - 2t_o n \quad ,$$

$$p + 2q \quad ,$$

$$p + 2q \quad ,$$

$$\vdots$$

$$p + 2q \quad ,$$

$$\frac{i^2}{d} + 1 - \frac{i^2}{d} + q = q + 1 \quad , \text{ respectively.}$$

Multiplying these totals by $\frac{d}{p + 2q}$ and adding to column 1 gives

$$|\underline{B}| = \begin{vmatrix} 1 + \frac{dn}{p+2q} & \frac{f}{d} & 1 & 1 & 1 & \cdots & 1 & 1 & \frac{i}{d} \\ -f + \frac{d(q+1-2t_o n)}{p+2q} & a-\frac{f^2}{d} & q & 0 & 0 & \cdots & 0 & 0 & \frac{i^2}{d} \\ 0 & q & p & q & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & p & q & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & q & p & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & q & p & q \\ -i + \frac{d(q+1)}{p+2q} & \frac{i^2}{d} & 0 & 0 & 0 & \cdots & 0 & q & 1-\frac{i^2}{d} \end{vmatrix} .$$

The column totals from row 2 to row n+2 inclusively are

$$- (i + f) + \frac{d}{p + 2q}(2q + 2 - 2t_o n) = \frac{d(1 - \beta^2)}{p + 2q} ,$$

$$q + a - \frac{f^2}{d} + \frac{i^2}{d} = q + 1 - 2t_o n ,$$

$$p + 2q ,$$

$$\vdots$$

$$p + 2q ,$$

$$\frac{i^2}{d} + q + 1 - \frac{i^2}{d} = q + 1 , \quad \text{respectively.}$$

Multiplying these totals by $\frac{-1}{p + 2q}$ and adding to row 1 gives

$$|\underline{B}| = \begin{vmatrix} h & j & 0 & 0 & 0 & \cdots & 0 & 0 & k \\ l & a-\frac{f^2}{d} & q & 0 & 0 & \cdots & 0 & 0 & \frac{i^2}{d} \\ 0 & q & p & q & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & p & q & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & q & p & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & q & p & q \\ m & \frac{i^2}{d} & 0 & 0 & 0 & \cdots & 0 & q & 1-\frac{i^2}{d} \end{vmatrix} ,$$

where

$$h = 1 + \frac{dn}{p + 2q} - \frac{d(1 - \beta^2)}{(p + 2q)^2} ,$$

$$j = \frac{f}{d} - \frac{(q + 1 - 2t_o n)}{p + 2q} ,$$

$$k = \frac{i}{d} - \frac{(q + 1)}{p + 2q} ,$$

$$l = - f + \frac{d(q + 1 - 2t_o n)}{p + 2q} ,$$

$$m = - i + \frac{d(q + 1)}{p + 2q} .$$

Multiplying column 1 by $\frac{-i^2}{dm}$ and adding to column 2 and then multiplying row 1 by $\frac{-i^2}{dk}$ and adding to row 2 gives

$$|\underline{B}| = \begin{vmatrix} h & j-\frac{i^2h}{dm} & 0 & 0 & 0 & \cdots & 0 & 0 & k \\ l-\frac{i^2h}{dk} & a-\frac{f^2}{d}-\frac{i^2l}{dm}-\frac{i^2}{dk}(j-\frac{i^2h}{dm}) & q & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & p & q & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & p & q & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & q & p & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & q & p & q \\ m & 0 & 0 & 0 & 0 & \cdots & 0 & q & 1-\frac{i^2}{d} \end{vmatrix} .$$

Letting

$$s = j - \frac{i^2h}{dm} ,$$

$$w = l - \frac{i^2h}{dk} ,$$

$$x = a - \frac{f^2}{d} - \frac{i^2l}{dm} - \frac{i^2}{dk}(j - \frac{i^2h}{dm}) ,$$

$$y = 1 - \frac{i^2}{d} ,$$

we have

$$|\underline{B}| = \begin{vmatrix} h & s & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & k \\ w & x & q & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & p & q & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & p & q & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & q & p & q & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & p & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & p & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & q & p & q \\ m & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & q & y \end{vmatrix}_{(n+2)\times(n+2)} .$$

Note that

$$m = -\,dk \quad ,$$

$$l = -\,dj \quad ,$$

$$w = -\,ds \quad .$$

Expanding $|\underline{B}|$ along the first column gives

$$(2.4.3) \qquad |\underline{B}| = h\cdot Q_{n+1} - w\cdot M_{n+1} + (-1)^{n+1} m\cdot N^{n+1} \quad ,$$

where Q_{n+1} , M_{n+1} , and N_{n+1} are defined by

$$Q_{n+1} = \begin{vmatrix} x & q & 0 & 0 & & & \\ q & p & q & 0 & & (0) & \\ 0 & q & p & q & & & \\ 0 & 0 & q & p & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & p & q & 0 \\ & (0) & & & q & p & q \\ & & & & 0 & q & y \end{vmatrix}_{n+1} \quad ,$$

$$M_{n+1} = \begin{vmatrix} s & 0 & 0 & 0 & \cdots & 0 & 0 & k \\ q & p & q & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & p & q & \cdots & 0 & 0 & 0 \\ 0 & 0 & q & p & \cdots & 0 & 0 & 0 \\ & & & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & & \ddots & p & q & 0 \\ & (0) & & & & q & p & q \\ & & & & & 0 & q & y \end{vmatrix}_{n+1} \quad ,$$

$$N_{n+1} = \begin{vmatrix} s & 0 & 0 & 0 & \cdots & 0 & 0 & k \\ x & q & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & p & q & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & p & q & \cdots & 0 & 0 & 0 \\ & & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & & \ddots & q & 0 & 0 \\ & (0) & & & & p & q & 0 \\ & & & & & q & p & q \end{vmatrix}_{n+1} .$$

Expanding M_{n+1} and N_{n+1} along the elements of the first row gives

$$M_{n+1} = s \cdot X_n + (-1)^n \cdot k \cdot q^n \qquad \text{, and}$$

$$N_{n+1} = s \cdot q^n + (-1)^n \cdot k \cdot Y_n \qquad \text{respectively,}$$

where

$$X_n = \begin{vmatrix} p & q & 0 & & & \\ q & p & q & \ddots & (0) & \\ 0 & q & p & \ddots & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & p & q & 0 \\ & (0) & & q & p & q \\ & & & 0 & q & y \end{vmatrix} ,$$

$$Y_n = \begin{vmatrix} x & q & 0 & & & \\ q & p & q & \ddots & (0) & \\ 0 & q & p & \ddots & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & p & q & 0 \\ & (0) & & q & p & q \\ & & & 0 & q & p \end{vmatrix} .$$

We have used the fact that the determinant of a triangular matrix is the product of the diagonal entries.

Therefore

$$|\underline{B}| = hQ_{n+1} - w\left(sX_n + (-1)^n kq^n\right) + (-1)^{n+1} m\left(sq^n + (-1)^n kY_n\right)$$

$$= hQ_{n+1} - swX_n - kmY_n + (-1)^{n+1}(sm + kw)q^n \quad .$$

Note also that

$$Q_{n+1} = xX_n - q^2 X_{n-1} \quad ,$$

so

$$(2.4.4) \qquad |\underline{B}| = h(xX_n - q^2 X_{n-1}) - swX_n - kmY_n + (-1)^{n+1}(sm + kw)q^n$$

$$= (-1)^{n+1}(sm + kw)q^n + (hx - sw)X_n - kmY_n - q^2 hX_{n-1} \quad .$$

Next, we calculate X_n and Y_n.

It is easily seen that $X_1 = y$, $X_2 = py - q^2$, and by expanding along the first row

$$X_n = pX_{n-1} - q^2 X_{n-2} \quad .$$

Therefore

$$(2.4.5) \qquad X_{n+2} - pX_{n+1} + q^2 X_n = 0 \quad .$$

Define the operator E by $EX_k = X_{k+1}$. Then we may write equation (2.4.5) as

$$(2.4.6) \qquad (E^2 - pE + q^2)X_n = 0 \quad .$$

The roots of this equation are

$$u = \frac{p + \sqrt{p^2-4q^2}}{2} \qquad \text{and} \qquad v = \frac{p - \sqrt{p^2-4q^2}}{2}$$

where $u + v = p$ and $u \cdot v = q^2$.

Then the general solution of equation (2.4.6) is

$$(2.4.7) \qquad X_n = c_1u^n + c_2v^n \quad .$$

Since $X_1 = y$ and $X_2 = py - q^2$, we have

$$X_1 = c_1u + c_2v = y \qquad \text{and}$$

$$X_2 = c_1u^2 + c_2v^2 = py - q^2 \quad .$$

Solving for c_1 and c_2 gives

$$c_1 = \frac{y(p - v) - q^2}{u(u - v)} \qquad \text{and}$$

$$c_2 = \frac{y(u - p) + q^2}{v(u - v)} \quad .$$

Thus, (2.4.7) becomes

$$(2.4.8) \qquad X_n = \left[\left(\frac{y(p - v) - q^2}{u(u - v)}\right)u^n + \left(\frac{y(u - p) + q^2}{v(u - v)}\right)v^n\right] \quad .$$

Comparing X_n with Y_n gives

$$(2.4.9) \qquad Y_n = \left[\left(\frac{x(p - v) - q^2}{u(u - v)}\right)u^n + \left(\frac{x(u - p) + q^2}{v(u - v)}\right)v^n\right] \quad .$$

Substituting X_n and Y_n into $|\underline{B}|$ gives

$$\begin{aligned} |\underline{B}| &= (-1)^{n+1}(ms + kw)q^n \\ &\quad + (hx - sw)\left[\left(\frac{y(p - v) - q^2}{u(u - v)}\right)u^n + \left(\frac{y(u - p) + q^2}{v(u - v)}\right)v^n\right] \\ &\quad - km\left[\left(\frac{x(p - v) - q^2}{u(u - v)}\right)u^n + \left(\frac{x(u - p) + q^2}{v(u - v)}\right)v^n\right] \\ &\quad - q^2h\left[\left(\frac{y(p - v) - q^2}{u(u - v)}\right)u^{n-1} + \left(\frac{y(u - p) + q^2}{v(u - v)}\right)v^{n-1}\right] \quad . \end{aligned}$$

Now

$$ms + kw = -2dks$$

$$hx - sw = \frac{h(ad - f^2) + l^2}{d} ,$$

thus

(2.4.10) $$|\underline{B}| = (-1)^n(2dks)q^n$$

$$+ \left(\frac{h(ad - f^2) + l^2}{d}\right)\left(\frac{(y(p - v) - q^2)u^{n-1} + (y(u - p) + q^2)v^{n-1}}{u - v}\right)$$

$$+ dk^2\left(\frac{(x(p - v) - q^2)}{u - v}u^{n-1} + \frac{(x(u - p) + q^2)}{u - v}v^{n-1}\right)$$

$$- q^2h\left(\frac{(y(p - v) - q^2)}{u - v}u^{n-2} + \frac{(y(u - p) + q^2)}{u - v}v^{n-2}\right) .$$

Let $z + \frac{1}{z} = \frac{-p}{q}$.

Then $z^2 + 1 = \frac{-p}{q} z$ and $z^2 + \frac{p}{q} z + 1 = 0$.

Thus the roots are

$$z = \frac{-p + \sqrt{p^2 - 4q^2}}{2q} = \frac{-v}{q} \quad \text{and}$$

$$\frac{1}{z} = \frac{-(p + \sqrt{p^2 - 4q^2})}{2q} = \frac{-u}{q} .$$

Hence

(2.4.11) $u = \frac{-q}{z}$, $v = -qz$, and $u - v = \frac{-q}{z}(1 - z^2)$.

Replacing these values of u and v in $|\underline{B}|$, we obtain, after some simplification,

$$(2.4.12)\quad |\underline{B}| = (-1)^n(2dks)q^n + \frac{(-q)^{n-2}}{z^{n-2}(1-z^2)}\left\{\left(\frac{had - hf^2 + l^2}{d}\right)(yp + yqz - q^2) + dk^2(xp + xqz - q^2) + hqz(yp + yqz - q^2) + \left(\frac{had - hf^2 + l^2}{d}\right)(-yq - ypz + q^2z)z^{2n-5} + dk^2(-xq - xpz + q^2z)z^{2n-5} + hqz(-yq - ypz + q^2z)z^{2n-7}\right\} .$$

Ignoring the terms in z^{2n-5} and z^{2n-7}, since these give only an exponentially small error in the final approximation (see Patton (14), p. 26), we obtain the approximate result,

$$(2.4.13)\quad |\underline{B}| \simeq (-1)^n(2dks)q^n + \frac{(-q)^{n-2}}{z^{n-2}(1-z^2)}\left\{\left(\frac{had - hf^2 + l^2}{d} + hqz\right)(yp + yqz - q^2) + dk^2(xp + xqz - q^2)\right\}$$

$$= \frac{(-q)^{n-2}}{z^{n-2}(1-z^2)}\left\{2dksz^{n-2}(1-z^2)q^2 + dk^2(xp + xqz - q^2) + \left(\frac{had - hf^2 + l^2}{d} + hqz\right)(yp + yqz - q^2)\right\} .$$

Since, as we shall see later (p.39), $|z| < 1$, except in exceptional cases when $|z| = 1$, the first term is exponentially small with respect to the second so it can be neglected.

Therefore

$$(2.4.14)\qquad |\underline{B}| \simeq \frac{(-q)^{n-2}}{z^{n-2}(1-z^2)}\left\{\left(\frac{had - hf^2 + l^2}{d} + hqz\right)(yp + yqz - q^2) + dk^2(xp + xqz - q^2)\right\} .$$

Let the term in brackets be denoted by $\phi_n(z,u)$ and let $u = t_o + rt$. We want to express $|\underline{B}|$ in terms of z and u.

To do this, we need to simplify $\phi_n(z,u)$. It will suffice to consider $\phi_n(z,0)$, that is, at $u = 0$, since we will be evaluating

$$\frac{\partial}{\partial u} M(u - rt, t)\ \frac{\partial t}{\partial z}$$

at $u = 0$. (See the discussion on p. 40.)

Using $u = t_o + rt$ and $z + \frac{1}{z} = -\frac{p}{q}$, we find

$$(2.4.15)\qquad -q = \frac{z(1 + \beta^2 - 2\beta r - 2nu)}{1 - 2rz + z^2} .$$

Let

$$K(u) = \frac{(1 + \beta^2 - 2\beta r - 2nu)}{1 - 2rz + z^2} .$$

Then

$$q = -zK ,$$

and we obtain the following results:

$$t = \frac{zK - \beta}{n} ,$$

$$t_o = \frac{nu - r(zK - \beta)}{n} ,$$

$$a = 1 - 2(n - 1)\left(u - \frac{r(zK - \beta)}{n}\right) ,$$

$$d = 2\left\{u + (1 - r)\left(\frac{zK - \beta}{n}\right)\right\} ,$$

$$f = 2u + (1 - 2r)\left(\frac{zK - \beta}{n}\right) ,$$

$$i = \frac{zK - \beta}{n} .$$

Let

$$A = zK(0) - \beta = \frac{z(1 + \beta^2) - \beta(1 + z^2)}{1 - 2rz + z^2} .$$

Then, evaluating the above quantities at $u = 0$ gives:

$$a(0) = 1 + 2r\left(\frac{n - 1}{n}\right)(zK - \beta) = (1 + 2rA)(1 + O(n^{-1})) ,$$

$$d(0) = \frac{2(1 - r)A}{n} ,$$

$$f(0) = \frac{(1 - 2r)A}{n} ,$$

$$i(0) = \frac{A}{n} ,$$

$$q(0) = -(A + \beta) ,$$

$$t(0) = \frac{A}{n} ,$$

$$t_o(0) = -\frac{rA}{n} .$$

Substituting these values into $p(0)$, $h(0)$, $j(0)$, and $l(0)$ gives:

$$p(0) = 1 + \beta^2 + 2rA \quad ,$$

$$p(0) + 2q(0) = (1 - \beta)^2 - 2A(1 - r) \quad ,$$

$$h(0) = 1 + \frac{2(1 - r)A}{(1 - \beta)^2 - 2A(1 - r)} - \frac{2(1 - r)A(1 - \beta^2)}{n\{(1 - \beta)^2 - 2A(1 - r)\}^2}$$

$$= \left\{1 + \frac{2(1 - r)A}{(1 - \beta)^2 - 2A(1 - r)}\right\}(1 + O(n^{-1})) \quad ,$$

$$j(0) = \frac{(1 - 2r)}{2(1 - r)} - \frac{(1 - \beta) - A(1 - 2r)}{(1 - \beta)^2 - 2A(1 - r)} \quad ,$$

$$l(0) = \frac{2(1 - r)A\{(1 - \beta) - A(1 - 2r)\}}{n\{(1 - \beta)^2 - 2A(1 - r)\}} - \frac{A(1 - 2r)}{n} = O(n^{-1}) \quad .$$

Thus

$$h(0)a(0)d(0) = \left\{1 + \frac{2(1 - r)A}{(1 - \beta)^2 - 2A(1 - r)}\right\}(1 + 2rA)$$

$$\times \left\{\frac{2(1 - r)A}{n}\right\}(1 + O(n^{-1})) \quad ,$$

$$h(0)f^2(0) = \left\{1 + \frac{2(1 - r)A}{(1 - \beta)^2 - 2A(1 - r)}\right\}\frac{(1 - 2r)^2A^2}{n^2}(1 + O(n^{-1})),$$

$$l^2(0) = O(n^{-2}) \quad .$$

Therefore, since $h(0)f^2(0)$ and $l^2(0)$ are small compared with $h(0)a(0)d(0)$, we have

$$\frac{h(0)a(0)d(0) - h(0)f^2(0) + l^2(0)}{d(0)}$$

$$= \left\{1 + \frac{2(1 - r)A}{(1 - \beta)^2 - 2A(1 - r)}\right\}(1 + 2rA)(1 + O(n^{-1})) \quad .$$

Now,

$$q(0)h(0)z = -(A + \beta)\left\{1 + \frac{2(1 - r)A}{(1 - \beta)^2 - 2A(1 - r)}\right\}z(1 + O(n^{-1})) \quad .$$

Therefore,

$$(2.4.16) \quad \frac{h(0)a(0)d(0) - h(0)f^2(0) + l^2(0)}{d(0)} + q(0)h(0)z$$

$$= \left\{1 + \frac{2(1 - r)A}{(1 - \beta)^2 - 2A(1 - r)}\right\}\left(1 - z\beta - A(z - 2r)\right)(1 + O(n^{-1})) \quad .$$

To find $y(0)p(0) + y(0)q(0)z - q^2(0)$, observe that

$$y(0)p(0) = \left\{1 - \frac{A}{2n(1 - r)}\right\}(1 + \beta^2 + 2rA) \simeq 1 + \beta^2 + 2rA \quad ,$$

$$y(0)q(0)z = -\left\{1 - \frac{A}{2n(1 - r)}\right\}(A + \beta)z \simeq -(A + \beta)z \quad ,$$

$$q^2(0) = (A + \beta)^2 \quad .$$

Thus

$$(2.4.17) \quad y(0)p(0) + y(0)q(0)z - q^2(0)$$

$$= \left\{1 - \frac{A}{2n(1 - r)}\right\}\left(1 - \beta z + \beta^2 + A(2r - z)\right) - (A + \beta)^2.$$

Thus the first term of $\phi_n(z,0)$ is approximately equal to

$$(2.4.18) \quad \left\{1 + \frac{2(1-r)A}{(1-\beta)^2 - 2A(1-r)}\right\}\left\{1 - z\beta + A(2r - z)\right\}$$

$$\times \left[\left\{1 - \frac{A}{2n(1-r)}\right\}\left\{1 - z\beta + \beta^2 + A(2r - z)\right\} - (A + \beta)^2\right]$$

where the relative error is $O(n^{-1})$.

In order to express this in terms of β, r and z, we need the following results:

$$1 - z\beta + A(2r - z) = \frac{(1 - \beta z)(1 + \beta z - 2r\beta)}{1 + z^2 - 2rz} ,$$

$$1 - z\beta + \beta^2 + A(2r - z) = \frac{1 - 2r\beta + \beta^2}{1 + z^2 - 2rz} ,$$

$$(1 - \beta)^2 - 2A(1 - r) = \frac{(1 - z)^2(1 + \beta^2 - 2r\beta)}{1 + z^2 - 2rz} ,$$

$$1 + \frac{2(1-r)A}{(1-\beta)^2 - 2A(1-r)} = \frac{(1-\beta)^2(1 + z^2 - 2rz)}{(1-z)^2(1 + \beta^2 - 2r\beta)} .$$

Then,

$$(2.4.19) \quad \left\{1 + \frac{2(1-r)A}{(1-\beta)^2 - 2A(1-r)}\right\}\left\{1 - z\beta + A(2r - z)\right\}$$

$$= \frac{(1-\beta)^2(1 - \beta z)(1 + \beta z - 2r\beta)}{(1-z)^2(1 + \beta^2 - 2r\beta)} ,$$

and

$$\left(1 - \frac{A}{2n(1 - r)}\right)\left(1 - \beta z + \beta^2 + A(2r - z)\right) - (A + \beta)^2 \tag{2.4.20}$$

$$= \left(1 - \beta z + \beta^2 + A(2r - z)\right) - (A + \beta)^2 + O(n^{-1})$$

$$= \frac{(1 + \beta^2 - 2r\beta)(1 - \beta z)(1 + \beta z - 2rz)}{(1 + z^2 - 2rz)^2} + O(n^{-1}) \quad .$$

By equations (2.4.19) and (2.4.20), the first term of $\phi_n(z,0)$,

$$\left(\frac{h(0)a(0)d(0) - h(0)f^2(0) + l^2(0)}{d(0)} + q(0)h(0)z\right) \tag{2.4.21}$$

$$\times \left(y(0)p(0) + y(0)q(0)z - q^2(0)\right)$$

$$= \frac{(1 - \beta)^2(1 - \beta z)^2(1 + \beta z - 2r\beta)(1 + \beta z - 2rz)}{(1 - z)^2(1 + z^2 - 2rz)^2}\left(1 + O(n^{-1})\right) \quad .$$

Consider now the second term of $\phi_n(z,0)$; that is,

$$dk^2(xp + xpq - q^2)$$

evaluated at $u = 0$.

We have

$$m(0) = -\frac{A}{n} + \frac{2(1 - r)A(1 - A - \beta)}{n\left((1 - \beta)^2 - 2A(1 - r)\right)}$$

$$= \frac{(1 - \beta)(2r + \beta - 1)\left(-z(1 + \beta^2) + \beta(1 + z^2)\right)}{n(1 - z^2)(1 - 2r\beta + \beta^2)} \quad ,$$

$$k(0) = -\frac{m(0)}{d(0)} \quad .$$

Therefore

$$d(0)k^2(0) = \frac{m^2(0)}{d(0)}$$

$$= \frac{(1-\beta)^2(2r+\beta-1)^2(1-2rz+z^2)\{z(1+\beta^2) - \beta(1+z^2)\}}{2n(1-r)(1-z^2)^2(1-2r\beta+\beta^2)^2}$$

$$= O(n^{-1}) \quad .$$

Also

$$x(0) = a(0) - \frac{f^2(0)}{d(0)} - \frac{2i^2(0)l^2(0)}{d(0)m(0)} - \frac{i^4(0)h(0)}{d(0)m^2(0)}$$

$$= 1 + 2rA - \frac{(1-2r)^2}{4(1-r)^2}$$

$$+ \frac{A}{(1-r)} \frac{2(1-r)\{(1-\beta)-A(1-2r)\} - (1-2r)\{(1-\beta)^2-2A(1-r)\}}{(1-\beta)^2 - 2(1-r)(1-\beta)}$$

$$\times (1 + O(n^{-1})) \quad .$$

Since

$$p(0) = 1 + \beta^2 + 2rA \qquad \text{and}$$

$$q(0) = -(A + \beta) \qquad ,$$

$x(0)p(0) + x(0)p(0)q(0) - q^2(0)$ will be of the same order as $x(0)$.

Now,

$$d(0)k^2(0) = O(n^{-1})$$

so

$$d(0)k^2(0)\{x(0)p(0) + x(0)p(0)q(0) - q^2(0)\} = O(n^{-1}) \tag{2.4.22}$$

and is therefore small compared with the first term of $\phi_n(z,0)$.

Thus

$$\phi_n(z,0) = \frac{(1-\beta)^2(1-\beta z)^2(1+\beta z-2r\beta)(1+\beta z-2rz)}{(1-z)^2(1+z^2-2rz)^2}(1 + O(n^{-1})) \quad . \tag{2.4.23}$$

Recall from equations (2.4.14) and (2.4.15) that

$$|\underline{B}| = \frac{(-q)^{n-2}\phi_n(z,u)}{z^{n-2}(1-z^2)} = \frac{(1-2r\beta+\beta^2-2nu)^{n-2}}{(1-z^2)(1-2rz+z^2)^{n-2}}\phi_n(z,u)$$

and

$$\beta + tn = \frac{z(1-2r\beta+\beta^2-2nu)}{1-2rz+z^2} .$$

Then

$$t = \frac{z(1-2r\beta+\beta^2-2nu)}{n(1-2rz+z^2)} - \frac{\beta}{n} ,$$

so that

(2.4.24) $$\frac{\partial t}{\partial z} = \frac{(1-2r\beta+\beta^2-2nu)(1-z^2)}{n(1-2rz+z^2)^2} .$$

The joint moment generating function $M(t_o,t)$ was given by (2.4.1),

$$M(t_o,t) = \frac{(1-\beta^2)^{\frac{1}{2}}}{\left\||\underline{B}|\right\|^{\frac{1}{2}}} .$$

Substituting for $|\underline{B}|$ and writing t_o as $u - rt$ gives

$$M(u - rt,t) = \frac{(1-\beta^2)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n-2}{2}}}{(1-2r\beta+\beta^2-2nu)^{\frac{n-2}{2}}\,|\phi_n(z,u)|^{\frac{1}{2}}} .$$

Multiplying this by $\frac{\partial t}{\partial z}$ and combining terms gives

$$M(u - rt,t)\,\frac{\partial t}{\partial z} = \frac{(1-\beta^2)^{\frac{1}{2}}(1-z^2)^{\frac{3}{2}}\,(1-2rz+z^2)^{\frac{n}{2}-3}}{n(1-2r\beta+\beta^2-2nu)^{\frac{n}{2}-2}\,|\phi_n(z,u)|^{\frac{1}{2}}}$$

The contribution of the partial derivative of $\phi_n(z,u)$ with respect to u is so small compared with the partial derivative of $(1-2r\beta+\beta^2-2nu)^{\frac{1}{2}n-2}$ that we can evaluate $\phi_n(z,u)$ at $u = 0$ prior

to taking the partial derivatives.

Thus,

$$(2.4.25)\quad \frac{\partial}{\partial u}\left\{M(u - rt,t)\ \frac{\partial t}{\partial z}\right\}\Bigg|_{u=0}$$

$$= \frac{(1-\beta^2)^{\frac{1}{2}}(1-z^2)^{\frac{3}{2}}\ (1-2rz+z^2)^{\frac{n}{2}-3}\ (n-4)}{(1-2r\beta+\beta^2)^{\frac{n}{2}-1}\ |\phi_n(z,0)|^{\frac{1}{2}}} \quad .$$

Next, substitute the value of $\phi_n(z,0)$ into equation (2.4.25), giving,

$$(2.4.26)\quad \frac{\partial}{\partial u}\left\{M(u - rt,t)\ \frac{\partial t}{\partial z}\right\}\Bigg|_{u=0}$$

$$= \frac{n(1-z)(1-z^2)^{\frac{3}{2}}\ (1-\beta^2)^{\frac{1}{2}}(1-2rz+z^2)^{\frac{n}{2}-2}}{(1-\beta)(1-\beta z)(1+\beta z-2r\beta)^{\frac{1}{2}}|1+\beta z-2rz|^{\frac{1}{2}}(1-2r\beta+\beta^2)^{\frac{n}{2}-1}}(1 + O(n^{-1})) \quad .$$

Note that in equation (2.4.26), the factor $(n-4)$ has been replaced by n since $n-4 = n(1 + O(n^{-1}))$.

Recall from equation (2.1.5) that the probability density function of r, $h(r)$, can be written as

$$h(r) = \frac{1}{2\pi i}\int \frac{\partial}{\partial u}\left\{M(u - rt,t)\ \frac{\partial t}{\partial z}\right\}\Bigg|_{u=0} dz$$

where the integration is performed along the transformed contour in the z-plane.

The integrand may be written in the form $\theta(z)\left(\Psi(z)\right)^n$ so that

$$(2.4.27)\quad h(r) \simeq \frac{n(1-\beta^2)^{\frac{1}{2}}}{2\pi i(1-\beta)(1-2r\beta+\beta^2)^{\frac{n}{2}-1}}\int \theta(z)(1-2rz+z^2)^{\frac{n}{2}-2}$$

where

$$(2.4.28) \quad \theta(z) = \frac{(1-z)(1-z^2)^{\frac{3}{2}}}{(1-\beta z)(1+\beta z-2r\beta)^{\frac{1}{2}}|1+\beta z-2rz|^{\frac{1}{2}}} \quad .$$

The transformation from the t-plane to the z-plane defined by

$$z + \frac{1}{z} = \frac{1 + \beta^2 + 2rtn}{\beta + tn}$$

can be expressed as a series of elementary mappings. Note that

$$(2.4.29) \quad z + \frac{1}{z} = \frac{1 + \beta^2 - 2r\beta + 2r\beta + 2rtn}{\beta + tn} = \frac{1 + \beta^2 - 2r\beta}{\beta + tn} + 2r \quad .$$

Then the elementary mappings are:

A: $\quad V = \beta + tn$

B: $\quad U = \dfrac{V}{1 + \beta^2 - 2r\beta}$

C: $\quad S = \dfrac{1}{2U}$

D: $\quad Q = S + r$

E: $\quad z + \dfrac{1}{z} = 2Q \quad .$

E maps the Q-plane cut along the real axis from -1 to 1 to the interior of the unit circle in the z-plane. The other mappings change the endpoints of the cut on the real axis. So the transformation maps the whole t-plane cut along the real axis exterior to

$$\left(\frac{-(1 + \beta)^2}{2n(1 + r)} , \frac{(1 - \beta)^2}{2n(1 - r)}\right)$$

onto the interior of the unit circle.

The endpoints of the path of integration in the z-plane are found as follows.

As $t \to \pm i\infty$, $\left. z + \frac{1}{z} \right|_{u=0} = \frac{1 + \beta^2 - 2r\beta}{\beta + tn} + 2r \to 2r$;

then

$$z + \frac{1}{z} = 2r \quad \text{or} \quad z^2 - 2rz + 1 = 0 \ .$$

Thus,

$$z = \frac{2r \pm \sqrt{4r^2 - 4}}{2} = r \pm \sqrt{r^2 - 1} = r \pm i\sqrt{1 - r^2}$$

$$= e^{\pm i\theta} \quad , \quad \text{where} \quad r = \cos\theta \ .$$

If $t = 0$, by equation (2.4.29),

$$z + \frac{1}{z} = \frac{1 + \beta^2 - 2r\beta}{\beta} + 2r = \beta + \frac{1}{\beta} \ .$$

Therefore, in the z-plane, the path cuts the real axis at $z = \beta$. So the path of integration crosses the real axis at $z = \beta$ and ends on the boundary of the unit circle at $e^{-i\theta}$ and $e^{i\theta}$.

Upon inspection of the integrand of $h(r)$, we find that the only possible singularities between r and β would occur if

$$1 + \beta z - 2r\beta = 0 \qquad \text{or} \qquad 1 + \beta z - 2rz = 0.$$

Consider the first case, that is, $1 + \beta z - 2r\beta = 0$.

(i) $r \leq z \leq \beta$, $r < \beta$

If $\beta > 0$,

$$1 + \beta z - 2r\beta \geq 1 + \beta r - 2\beta r = 1 - \beta r > 0.$$

If $\beta < 0$,

$$1 + \beta z - 2r\beta \geq 1 + \beta^2 - 2r\beta = 1 - 2r\beta + \beta^2 + r^2 - r^2$$

$$= (\beta - r)^2 + (1 - r^2) > 0.$$

(ii) $\beta \leq z \leq r, \quad \beta < r$

If $\beta > 0$,

$$1 + \beta z - 2r\beta \geq 1 + \beta^2 - 2r\beta = (\beta - r)^2 + (1 - r^2) > 0.$$

If $\beta < 0$,

$$1 + \beta z - 2r\beta \geq 1 + \beta r - 2\beta r = 1 - \beta r > 0.$$

(iii) $\beta = z = r$

$$1 + \beta^2 - 2r\beta = 1 + \beta r - 2r\beta = 1 - \beta r > 0.$$

Thus, no singularities arise from the term $(1 + \beta z - 2r\beta)$ for z between r and β.

Now consider the second case, that is, $1 + \beta z - 2rz = 0$. Solving for z gives

$$z = \frac{1}{2r - \beta} \quad ,$$

thus a singularity occurs between r and β if $z = \dfrac{1}{2r - \beta}$. In order to deform the path of integration we must restrict the range of values that r may take for a given β.

For values of z between r and β, $1 + \beta z - 2rz$ will be non-zero in each of the following regions:

$$\frac{1}{2r - \beta} > \beta \quad \text{or} \quad \frac{1}{2r - \beta} < r \quad \text{if} \quad r < \beta \quad ,$$

$$\frac{1}{2r - \beta} < \beta \quad \text{or} \quad \frac{1}{2r - \beta} > r \quad \text{if} \quad r > \beta \quad .$$

Consider each of these regions individually.

$$\text{(i)} \quad r < \beta, \quad \frac{1}{2r - \beta} > \beta$$

If $\beta > 0$, $\tfrac{1}{2}\beta < r < \beta$.

If $\beta < 0$, no values of r fall in the region.

$$\text{(ii)} \quad r < \beta, \quad \frac{1}{2r - \beta} < r$$

If $\beta > 0$, $\dfrac{\beta - \sqrt{\beta^2 + 8}}{4} < r < \tfrac{1}{2}\beta$.

If $\beta < 0$, $\dfrac{\beta - \sqrt{\beta^2 + 8}}{4} < r < \beta$.

$$\text{(iii)} \quad r > \beta, \quad \frac{1}{2r - \beta} < \beta$$

If $\beta > 0$, no values of r fall in the region.

If $\beta < 0$, $\beta < r < \tfrac{1}{2}\beta$.

$$\text{(iv)} \quad r > \beta, \quad \frac{1}{2r - \beta} > r$$

If $\beta > 0$, $\beta < r < \dfrac{\beta + \sqrt{\beta^2 + 8}}{4}$

If $\beta < 0$, $\tfrac{1}{2}\beta < r < \dfrac{\beta + \sqrt{\beta^2 + 8}}{4}$

Combining these results gives the allowable range of values for r, that is,

$$\frac{\beta - \sqrt{\beta^2 + 8}}{4} < r < \frac{\beta + \sqrt{\beta^2 + 8}}{4} \qquad .$$

Note that for z lying between r and β , and r falling in the above range, $1 + \beta z - 2rz > 0$.

Thus, for r in the region $\left(\frac{\beta - \sqrt{\beta^2+8}}{4}, \frac{\beta + \sqrt{\beta^2+8}}{4}\right)$, the path of integration in the z-plane may be deformed to be the straight line joining $e^{-i\theta}$ to $e^{i\theta}$ $(r = \cos\theta)$ and crossing the real axis at $z = r$. Thus the deformed path is the straight line joining $r - i\sqrt{1-r^2}$ and $r + i\sqrt{1-r^2}$ and so may be written as

$$z = r + iw\sqrt{1-r^2} \quad \text{where} \quad -1 \le w \le 1. \tag{2.4.30}$$

Recall from equations (2.4.27) and (2.4.28) that

$$h(r) = \frac{n(1-\beta^2)^{\frac{1}{2}}}{2\pi i(1-\beta)(1-2r\beta+\beta^2)^{\frac{n}{2}-1}} \int \theta(z)(1-2rz+z^2)^{\frac{n}{2}-2} \, (1 + O(n^{-1})) \quad , \tag{2.4.31}$$

where

$$\theta(z) = \frac{(1-z)(1-z^2)^{\frac{3}{2}}}{(1-\beta z)(1+\beta z-2r\beta)^{\frac{1}{2}}|1+\beta z-2rz|^{\frac{1}{2}}} \quad .$$

In terms of r and w,

$$1 - 2rz + z^2 = (1 - r^2)(1 - w^2)$$

and

$$dz = i\sqrt{1 - r^2}\, dw \quad .$$

Substituting into equation (2.4.31) gives

$$h(r) = \frac{n(1-\beta^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\pi(1-\beta)(1-2r\beta+\beta^2)^{\frac{n}{2}-1}} \int_{-1}^{1} \theta(z)(1-w^2)^{\frac{n}{2}-2}\, dw \,(1 + O(n^{-1})) \quad .$$

Expand $\theta(z)$ as a power series in $(z - r)$ using equation (2.4.30).

$$\theta(z) = \theta(r) + \theta'(r)(z - r) + \ldots + \frac{\theta^{(k)}(r)(z - r)^k}{k!} + \ldots$$

$$= \theta(r) + iw\sqrt{1-r^2}\,\theta'(r) + \ldots + \frac{i^k w^k (1-r^2)^{\frac{k}{2}}}{k!}\,\theta^{(k)}(r) + \ldots$$

Thus,

$$h(r) = \frac{n(1-\beta^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\pi(1-\beta)(1-2r\beta+\beta^2)^{\frac{n}{2}-1}} \tag{2.4.32}$$

$$\times \sum_{k=0}^{\infty} \frac{i^k(1-r^2)^{\frac{k}{2}}}{k!}\,\theta^{(k)}(r) \int_{-1}^{1} w^k(1-w^2)^{\frac{n}{2}-2}\, dw$$

where the relative error is $O(n^{-1})$.

For $k = 2m + 1$, $m = 0,1,2,\ldots$, $w^k(1-w^2)^{\frac{n}{2}-2}$ is an odd function so the integral vanishes, giving

$$(2.4.33)\quad h(r) = \frac{n(1-\beta^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\pi(1-\beta)(1-2r\beta+\beta^2)^{\frac{n}{2}-1}}$$

$$\times \sum_{k=0}^{\infty} \frac{(-1)^k(1-r^2)^k}{(2k)!}\, \theta^{(2k)}(r) \int_{-1}^{1} w^{2k}(1-w^2)^{\frac{n}{2}-2}\, dw$$

$$\times\ (1 + O(n^{-1})) \quad .$$

Let

$$(2.4.34)\quad I_k = \int_{-1}^{1} w^{2k}(1-w^2)^{\frac{n}{2}-2}\, dw = 2\int_{0}^{1} w^{2k}(1-w^2)^{\frac{n}{2}-2}\, dw \quad ,$$

and let $v = w^2$, so that

$$dv = 2w\,dw \quad \text{and} \quad dw = \tfrac{1}{2}v^{-\frac{1}{2}}dv \quad .$$

Then

$$I_k = 2\int_{0}^{1} v^k(1-v)^{\frac{n}{2}-2} \cdot \tfrac{1}{2}v^{-\frac{1}{2}}dv$$

$$= \int_{0}^{1} v^{k-\frac{1}{2}}(1-v)^{\frac{n}{2}-2}\, dv \quad .$$

Compare I_k to the Beta function given by

$$B(m,n) = \int_{0}^{1} x^{m-1}(1-x)^{n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad .$$

Then

$$I_k = \int_{0}^{1} v^{(k+\frac{1}{2})-1}(1-v)^{(\frac{1}{2}n-1)-1}\, dv$$

$$= B(k + \frac{1}{2};\ \frac{n}{2} - 1)$$

$$= \frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{n}{2}-1)}{\Gamma(k+\frac{n}{2}-\frac{1}{2})} \quad .$$

Evaluate I_k for the first few values of k using the results

$$\Gamma(k+1) = k\Gamma(k) \quad \text{and} \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad .$$

We obtain,

$$\text{for } k = 0, \quad I_o = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} = \frac{\sqrt{\pi}\,\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \quad ;$$

$$k = 1, \quad I_1 = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n+1}{2})} = \frac{1}{n-1} I_o \quad ;$$

$$k = 2, \quad I_2 = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n+3}{2})} = \frac{1\cdot 3}{(n-1)(n+1)} I_o \quad .$$

In general, for $k = 1,2,\ldots$

$$(2.4.35) \quad I_k = \frac{1\cdot 3 \cdots (2k-1)}{(n-1)(n+1)\cdots(n+2k-3)} I_o$$

where

$$I_o = \frac{\sqrt{\pi}\,\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} \quad .$$

Using equation (2.4.34), equation (2.4.33) becomes

$$h(r) = \frac{n(1-\beta^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\pi(1-\beta)(1-2r\beta+\beta^2)^{\frac{n}{2}-1}} \sum_{k=0}^{\infty} \frac{(-1)^k(1-r^2)^k}{(2k)!} \theta^{(2k)}(r)\cdot I_k$$

$$\times (1 + 0(n^{-1})) \quad .$$

Substituting for I_k using equation (2.4.35) gives

$$h(r) = \frac{n(1-\beta^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\pi(1-\beta)(1-2r\beta+\beta^2)^{\frac{n}{2}-1}} \cdot I_o$$

$$\times \left\{\theta(r) + \sum_{k=1}^{\infty} \frac{(-1)^k(1-r^2)^k}{(2k)!}\,\theta^{(2k)}(r)\left[\frac{1\cdot 3\cdots(2k-1)}{(n-1)(n+1)\cdots(n+2k-3)}\right]\right\}$$

$$\times\ (1 + O(n^{-1}))$$

or,

$$(2.4.36)\quad h(r) = \frac{n\,\Gamma(\frac{n}{2}-1)(1-\beta^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}}{2\sqrt{\pi}\,\Gamma(\frac{n-1}{2})(1-\beta)(1-2r\beta+\beta^2)^{\frac{n}{2}-1}}$$

$$\times \left\{\theta(r) + \sum_{k=1}^{\infty} \frac{(-1)^k(1-r^2)^k\,\theta^{(2k)}(r)}{2^k k!\,((n-1)\cdots(n+2k-3))}\right\}(1 + O(n^{-1})) \quad .$$

Replacing z by r in equation (2.4.28) gives

$$\theta(r) = \frac{(1-r)(1-r^2)^{\frac{3}{2}}}{(1-\beta r)(1+\beta r-2r\beta)^{\frac{1}{2}}(1+\beta r-2r^2)^{\frac{1}{2}}}\ (1 + O(n^{-1}))$$

$$= \frac{(1-r)(1-r^2)^{\frac{3}{2}}}{(1-\beta r)^{\frac{3}{2}}(1+\beta r-2r^2)^{\frac{1}{2}}}\ (1 + O(n^{-1})) \quad .$$

Thus, a first approximation to $h(r)$ is

$$h(r) = \frac{n\,\Gamma(\frac{n}{2}-1)(1-\beta^2)^{\frac{1}{2}}(1-r^2)^{\frac{n-3}{2}}(1-r)(1-r^2)^{\frac{3}{2}}}{2\sqrt{\pi}\,\Gamma(\frac{n-1}{2})(1-\beta)(1-2r\beta+\beta^2)^{\frac{n}{2}-1}(1-\beta r)^{\frac{3}{2}}(1+\beta r-2r^2)^{\frac{1}{2}}}$$

$$\times\ (1 + O(n^{-1}))$$

$$(2.4.37) \quad = \frac{n\,\Gamma(\frac{n}{2}-1)(1-\beta^2)^{\frac{1}{2}}(1-r)(1-r^2)^{\frac{n}{2}}}{2\sqrt{\pi}\,\Gamma(\frac{n-1}{2})(1-\beta)(1-\beta r)^{\frac{3}{2}}(1+\beta r-2r^2)^{\frac{1}{2}}(1-2\beta r+\beta^2)^{\frac{n}{2}-1}}(1+O(n^{-1})) ,$$

where $\frac{\beta - \sqrt{\beta^2 + 8}}{4} < r < \frac{\beta + \sqrt{\beta^2 + 8}}{4}$.

CONCLUSION

In order to extend this result to all values of r between -1 and 1 it would be necessary to examine the behavior of the density at the singularity $z = \frac{1}{2r - \beta}$. The path of integration can not be deformed through this singularity so an alternative approach must be used. It can be shown that this singularity is a branch point and thus can be treated using techniques from complex analysis. The method would consist of choosing an appropriate contour around the branch point and adding up the contributions from each portion of the integral.

Note however that the major part of the probability lies within the allowable range of values for r . In fact at $r = \beta$ the probability is large and decreases rapidly on either side of β . As n increases, the peak becomes sharper and the tail probabilities decrease.

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